PARTIAL CONTROL AND AVOIDANCE OF ESCAPE FROM A POTENTIAL WELL

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Summary In this work, we present application of geometric and topological approaches of phase space transport to dynamical systems which exhibit transition out of a potential well or escape from the realm of bounded motion. This phenomena is observed in problems of celestial mechanics, chemical kinetics and ship dynamics where by transition and escape may be beneficial or detrimental. We present results in the context of mechanical systems that can be geometrically reduced to two-dimensional maps using invariant manifolds of unstable periodic orbits and suitable Poincaré surfaces-of-section. We apply a recently developed notion of controlling a dynamical system, in the presence of a random bounded disturbance while applying a smaller control, to avoid transition and escape into an undesirable realm of phase space.

PROBLEM DESCRIPTION

In a myriad of natural and engineering systems, there are instants of critical motion when the trajectory *escapes* from a (or *transitions* into another) metastable state, often characterized by a potential well. This phenomenon may be undesirable if it implies a catastrophic event or could be desirable from a control and design perspective. In any case, it is of paramount interest to understand the mechanisms underlying such critical motion, providing in essence a reduced order model of transitions. The discovery of such mechanisms will also lead us to propose strategies to avoid or trigger escape/transition and make better predictions.

We will illustrate our results in the context of a ship dynamics problem. Consider a model of ship dynamics that has nonlinear coupling of roll and pitch degrees of freedom (DOF) and is of interest in naval engineering for ship safety against capsize in the presence of wave forcing. The dynamical system of interest can be expressed by the Lagrangian $\mathcal{L}(x, y, \mathring{x}, \mathring{y})$ given by

$$\begin{aligned}\mathcal{L}(x, y, \mathring{x}, \mathring{y}) &= T(\mathring{x}, \mathring{y}) - V(x, y)\\ &= \frac{1}{2}\mathring{x}^2 + \frac{1}{2}\left(\frac{2}{R^2}\right)\mathring{y}^2 - \left(\frac{1}{2}x^2 + y^2 - x^2 y\right)\end{aligned} \qquad \left.\begin{aligned}\mathring{x} &= v_x\\ \mathring{y} &= v_y\\ \mathring{v}_x &= -x + 2xy + f_x(t)\\ \mathring{v}_y &= -R^2 y + \frac{1}{2}R^2 x^2 + f_y(t)\end{aligned}\right\} \quad (1)$$

with $V(x, y) = \frac{1}{2}x^2 + y^2 - x^2 y$ as the corresponding effective potential energy, x, y denotes the roll and pitch degrees of freedom which is non-dimensionalized using the roll and pitch angle of vanishing stability, $R = \omega_\theta/\omega_\phi$, ratio of the pitch to roll natural frequencies. In what follows, $R = 1.6$ is chosen so as to lessen the effects of parametric resonance. The equations of motion can be expressed in first order ODE form given by (1), where time is non-dimensionalized using the natural roll frequency and the non-conservative time-varying generalized forces, $f_x(t)$, $f_y(t)$, which denote rescaled angular accelerations due to wave moments. Our objective is to apply a geometrically motivated approach of identifying trajectories that lead to capsize and find if a control smaller than a disturbance can be used to avoid such event.

TUBE DYNAMICS AND PARTIAL CONTROL

When the system is autonomous i.e., ($f_x(t) = f_y(t) = 0$), Eqn. (1) conserves the energy, $E(x, y, v_x, v_y) = \frac{1}{2}v_x^2 + \frac{1}{2}\left(\frac{2}{R^2}\right)v_y^2 + \frac{1}{2}x^2 + y^2 - x^2 y$ since damping isn't considered and which represents a hypersurface in $\mathbb{R}^4$. However, by using a suitable geometric reduction technique we can classify orbits with varied fates for a given instantaneous energy, e. This is typically done by using a Poincaré surface-of-section (S-O-S), in this case a plane $\mathbb{R}^2$, that captures motion leading to escape from the potential well i.e., capsize. We consider a Poincaré S-O-S that is intersected by trajectories with motion to the right and given by $\Sigma_{U_1} = \{(y, v_y) | x = 0; v_x > 0\}$ (where $v_x > 0$ captures motion to the right) and shown in Fig. 1(a) and Fig. 1(b). Furthermore, the regions of energetically accessible motion for a ship of given energy, e, is defined by considering the projection of energy surface onto the configuration space, (x, y) plane, given by $\mathcal{M}(e) = \{(x, y) | V(x, y) \leqslant e\}$ which is historically known as *Hill's Region* (see [1]). Using basic dynamical systems theory, we obtain the critical points for the conservative system at $(\pm 1, 0.5, 0, 0)$ which is a rank-1 saddle (with eigenvalues $\pm\lambda, \pm i\omega$). The energy of saddle equilibrium points is defined as *critical energy* given by $E(\pm 1, 0.5, 0, 0) = E_{critical} = 0.25$ and all motions leading to escape from the potential well occur above this value i.e., a ship rolling and pitching with instantaneous energy e will capsize when $e > E_{critical}$.

This can be systematically explained by considering the invariant manifolds of the rank-1 saddles (this theory goes by the name of *tube dynamics* and is applied to celestial mechanics in [1]) to organize trajectories exhibiting ship's safe and capsize configuration. For a 2-DOF system (phase space is $\mathbb{R}^4$) the globalized manifolds are topologically a cylinder or *tubes* (hypersurface in $\mathbb{R}^4$ and co-dimension 1) and forms a boundary between capsize and non-capsize trajectories i.e., escape

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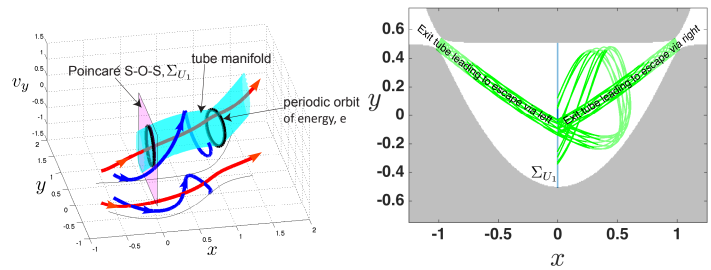
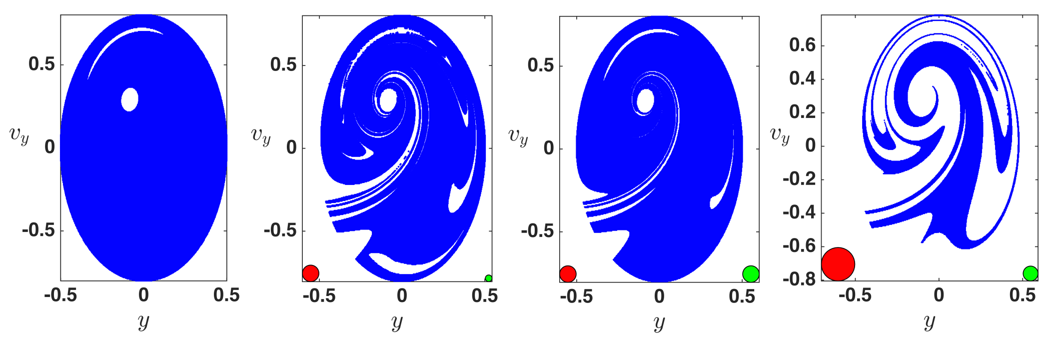


(a) (b)

Figure 1: Fig. 1(a) shows a schematic of escape (red) and non-escape (blue) trajectory, while tubes form the boundary for such critical motion and Poincaré S-O-S reduces the analysis to a study of map on $\mathbb{R}^2$. The boundary of Hills region i.e., zero velocity curve (kinetic energy vanishes) is shown in the (x, y) plane. Fig. 1(b) shows the Hill's region as shaded (where kinetic energy is negative) and the white regions are energetically accessible for $e > E_{critical}$. The stable tubes of the rank-1 saddle equilibrium points are shown in green which act as the boundary for escape and non-escape trajectories.

beyond the saddles of potential well as shown schematically in Fig. 1(a). Therefore dimension of *tubes* for a 2 DOF system is given by $\mathbb{S}^1 \times \mathbb{R}^1$ and thus its intersection with a Poincaré S-O-S, Σ_{U_1}, is $\mathbb{S}^1$ or 1-sphere in $\mathbb{R}^2$. On the plane, Σ_{U_1} the closed curves represent *forbibben* region (shown as white inside the blue in Fig. 2(a)) and trajectory entering this region will lead to imminent escape from the potential well. When the systems is non-autonomous, this geometric picture still forms the skeleton on which the escape manifests itself.

In presence of any disturbance, e.g., wave forcing or unmodeled fluid-vessel interaction, escape from the potential well becomes more prominent and our objective is to ask if it is possible to avoid escaping i.e., not entering the forbidden regions by exploiting the topology of *tubes* of the rank-1 saddles. This can be answered using a recently developed approach of *partial control* which is based on the notion of a safe set (see [2]). A safe set, say S, is a subset of the set we want to stay in, say Q, such that for every point $\mathbf{q} \in S$, we have $\max_{\mathbf{q}\in S, ||\boldsymbol{\xi}|| \leqslant \boldsymbol{\xi}_0} dist(f(\boldsymbol{q}) + \boldsymbol{\xi}, S) = \boldsymbol{u}_0 < \boldsymbol{\xi}_0$. When the safe set exists, we can find



(a) (b) (c) (d)

Figure 2: Fig. 2(a) shows the initial set in blue on the Poincaré S-O-S, Σ_{U_1} with white regions inside the blue denoting intersection of tube manifolds. Fig. 2(b), 2(c) and 2(d) show the safe sets in blue for $\xi_0 = 0.05, 0.05, 0.1$ for $u_0 = 0.405\xi_0, \xi_0, 0.445\xi_0$ respectively. The red and green disks denote the disturbance and control magnitudes that are applied on the map.

safety by using a control that is smaller than the disturbance. This is illustrated for an arbitrary disturbance, ξ_0, acting on the return map $\Sigma_{U_1} \to \Sigma_{U_1}$ in Fig. 2 and computed using the sculpting algorithm in [3].

CONCLUSION

We considered two arbitrary disturbance magnitudes, for illustration here, $\xi_0 = 0.05, 0.1$ and show that the safe sets exist for a minimum control magnitudes of $u_0 = 0.405\xi_0, 0.445\xi_0$ for an instantaneous energy, $e = 0.25307$. Thus, a ship's safety can be ensured by controlling the trajectories from entering the stable tubes that lead to imminent escape from the potential well. However, from a ship's motion stand point, the magnitude of a disturbance needs to be related to the wave forcing that acts in a rough/regular sea environment while interpreting the applied control magnitude in the form of a algorithmic control law.

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