ESCAPE FROM POTENTIAL WELLS IN MULTI-DIMENSIONAL EXPERIMENTAL SYSTEMS

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Summary Predicting the escape from a potential energy well is a universal exercise, governing myriad engineering and natural systems, e.g., buckling phenomena, ship capsize, and human balance. Criteria and routes of escape have previously been determined for 1 degree of freedom (DOF) mechanical systems with time-varying forcing, with reasonable agreement with experiments. When there are 2 or more DOF, the situation becomes more complicated, and the theory of tube dynamics provides the criteria for which phase space states will escape. We report the validation of the tube dynamics theory for a 2 DOF experiment of a ball rolling on a surface. This experimental validation establishes a theoretical framework which can be exploited for purposes of control, e.g., avoiding or triggering escape or transition between metastable states in mechanical systems.

INTRODUCTION

Our objective is to study routes of escape from a potential well in an experimental 2 degree of freedom system. For this aim we study the chaotic motion of a rolling ball on a surface ($H(x,y)$) which has 4 potential wells, one in each quadrant of the (x,y) plane, Fig. 1(a)(b) [1]. We adopt a global geometric view of the motion analysis, using techniques which have been fruitful in other areas of mechanics, such as celestial mechanics [2] and physical chemistry [3]. The equations of motion are obtained from the Lagrangian; $\mathcal{L}(x,y,\dot{x},\dot{y}) = T(x,y,\dot{x},\dot{y}) - U(x,y)$. The kinetic energy (translational plus rotational for a ball rolling without slipping) is,

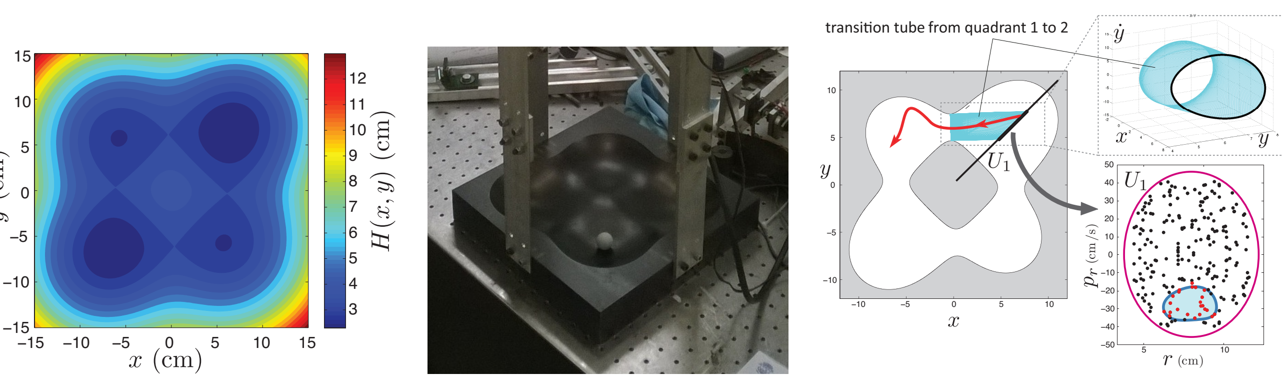
$$T(x,y,\dot{x},\dot{y}) = \frac{1}{2}\frac{7}{5}\left(\dot{x}^2 + \dot{y}^2 + (H_x\dot{x} + H_y\dot{y})^2\right) \tag{1}$$

and the potential energy is $U(x,y) = gH(x,y)$, where,

$$H(x,y) = \alpha(x^2+y^2) - \beta(\sqrt{x^2+\gamma} + \sqrt{y^2+\gamma}) - \xi xy + H_0. \tag{2}$$

We use parameter values $(\alpha,\beta,\gamma,\xi) = (0.07, 1.017, 15.103, 0.00656)$ in the appropriate units, along with $H_0 = 12.065$ cm and $g = 981$ cm/s^2. The ball's mass factors out and is not included.

Small damping is present, but over short time-scales, the motion approximately conserves energy, and the conservative dynamics are the dominant contributor to transition between wells. Let $\mathcal{M}$ be the *energy manifold* given by setting the energy integral ($\mathcal{E}(x,y,v_x,v_y) = T(x,y,\dot{x},\dot{y}) + U(x,y)$) equal to a constant, i.e., $\mathcal{M}(E) = \{(x,y,v_x,v_y) \subset \mathcal{R}^4 \mid \mathcal{E}(x,y,v_x,v_y) = E\}$ where E is a constant. The projection of the energy manifold onto configuration space, the (x,y) plane, is the region of



(a) (b) (c)

Figure 1: (a) Surface height $H(x,y)$. (b) Experimental apparatus. (c) For a fixed energy, E, above a critical value E_e, the permissible region (in white) has potential wells connected by necks around saddle-type equilibria. All motion from the well in quadrant 1 to quadrant 2 must occur through the interior of a stable manifold tube associated to an unstable periodic orbit in the neck between them. In the schematic of the U_1^+ Poincaré section, we the color intersecting trajectories by their near-future fate; black = no transition, red = imminent transition from quadrant 1 to 2 (an example of such a trajectory is also shown). A schematic tube boundary and interior is also shown for comparison on U_1^+. The large oval is the theoretical boundary of the energy manifold on U_1^+.

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energetically possible motion for a ball of energy E, $M(E) = \{(x, y) \mid U(x, y) \leq E\}$. The zero velocity curves are the boundary of $M(E)$ and are the locus of points in the (x, y) plane where the kinetic energy vanishes. The ball's state is only able to move on the side of this curve where the kinetic energy is positive, shown in white in Fig. 1(c). The critical energy of escape, E_e, is the same as the energy of the saddle points in each neck (which are all equal), and divides the global behavior into two cases, according to the sign of $\Delta E = E - E_e$:

Case 1, $\Delta E < 0$: the ball is safe against escape since potential wells are not energetically connected.

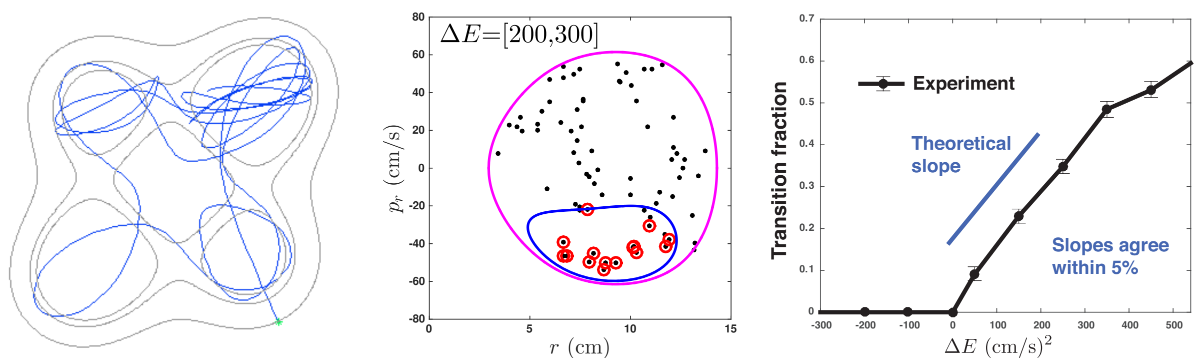
Case 2, $\Delta E > 0$: "necks" between all the potential wells open up around the saddle points, permitting the ball to move between the two potential wells (e.g., Fig. 1(c) shows this case).

TUBE DYNAMICS: TUBES LEADING TO ESCAPE

Within a given potential well the set of all states leading to escape to a different potential well (or having just escaped a different potential well) are within a cylindrical manifold or *tube*, as shown in Fig. 1(c). This tube bounds the set of all states for a fixed energy which will soon reach, or have just passed from, a different potential well [2, 3] (nested energy manifolds will have correspondingly nested tubes). For each E, the boundary of the tubes in phase space (or more precisely, within $\mathcal{M}(E)$) are the stable and unstable manifolds of an unstable periodic orbit of the same energy residing in the neck connecting the adjacent wells. The resulting geometric framework for understanding escape and transition we term 'tube dynamics'.

EXPERIMENTAL RESULTS

We use a Poincaré section which is selected based on the symmetry of the surface and the equations of motion, and which are best described in polar coordinates (r, p_r); $U_1^{\pm} = \{(r, p_r) \mid \theta = \frac{\pi}{2},\ \mathrm{sign}(p_\theta) = \pm 1\}$. Taking Poincaré sections of 120 experimental trajectories (e.g., the typical trial shown in Fig. 2(a)) should reveal the tube cross-sections. We first determine the instantaneous ΔE for every point on the Poincaré section U_1, so we can consider only narrow ranges of ΔE, to approximate a single energy manifold. In Fig. 2(b) we see an example of the Poincaré section U_1^+ for all intersections in the energy range $200 < \Delta E < 300\ (\mathrm{cm/s})^2$. The intersection points which are about to transition from quadrant 1 to 2, determined by following the experimental trajectory forward in time, and are marked with red circles.



(a) (b) (c)

Figure 2: *Experimental results.* (a) A typical experimental trajectory in blue, which is released from rest at the lower right. Notice how it moves seemingly erratically between the 'wells'. Some height iso-contours are also shown, in light gray. (b) Histogram of energy for crossings of the U_1^+ Poincaré section (blue: $\Delta E < 0$, gray: $\Delta E > 0$, red: transitioning). (c) On U_1^+, we consider only a narrow range of energy ($\Delta E \in [0, 100\ (\mathrm{cm/s})^2]$) and label intersecting trajectories by their recent past or future; black: no transition, red: recent transition to quadrant 1 from quadrant 2 and magenta: imminent transition from quadrant 1 to 4. (d) Fraction of transitioning trajectories as a function of energy above the saddle.

CONCLUSIONS

The transitioning points at each energy are all found to be within the theoretical tube boundary (blue curve); as in, e.g., Fig. 2(b). Furthermore, the fraction of transitioning trajectories increases linearly with ΔE, Fig. 2(c), as expected theoretically from arguments related to the phase space flux over a saddle [4]. This experimental validation of tube dynamics establishes a theoretical framework which can be exploited for purposes of control, e.g., avoiding or triggering escape or transition between metastable states in mechanical systems. SDR thanks the NSF for partially funding this work through grant 1537349.

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